REALISM IN MATHEMATICS
For
Dick and Steve
PREFACE

The philosophy of mathematics is a borderline discipline, of fundamental importance to both mathematics and philosophy. Despite this, one finds surprisingly little co-operation between philosophers and mathematicians engaged in its pursuit; more often, widespread disregard and misunderstanding are broken only by alarming pockets of outright antagonism. (The glib and dismissive formalism of many mathematicians is offset by the arrogance of those philosophers who suppose they can know what mathematical objects are without knowing what mathematics says they are.) This might not matter much in another age, but it does today, when the most pressing foundational problems are unlikely to be answered without a concerted co-operative effort. I have tried in this book to do justice to the concerns of both parties, to present the background, the issues, the proposed solutions on a neutral ground where the two sides can meet for productive debate.

For this reason, I’ve aimed for a presentation accessible to both non-philosophical mathematicians and non-mathematical philosophers and, if I’ve succeeded, students and interested amateurs should also be served. As far as I can judge, very little philosophical training or background is presupposed here. Mathematical prerequisites are more difficult to avoid, owing to the relentlessly cumulative nature of the discipline, but I’ve tried to keep them to a minimum. Some familiarity with the calculus and its foundations would be helpful, though surely not necessary. And the relevant set theoretic concepts are referenced to Enderton’s excellent introductory textbook (see his (1977)), for the benefit of those innocent of that subject.

The central theme of the book is the delineation and defence of a version of realism in mathematics called ‘set theoretic realism’. In this, my deep and obvious debt is to the writings of the great mathematical realists of our day: Kurt Gödel, W. V. O. Quine, and
Hilary Putnam (in the early 1970s). More personally, I have learned most from John Burgess, Paul Benacerraf, Hartry Field, and Tony Martin. After these, it would be impossible to mention everyone, but I can’t overlook the forceful criticisms of Charles Chihara, the insightful comments of Anil Gupta, and the generous correspondence, assistance, and advice of Philip Kitcher and Michael Resnik. Most recently, Burgess, Field, Lila Luce, Colin McLarty, Martin, Alan Nelson, Resnik, Stewart Shapiro, Mark Wilson, and Peter Woodruff have all done me the service of reading and reacting to drafts of various parts of the manuscript. (Naturally, the remaining errors and oversights should be charged to my shortcomings rather than to their negligence.) And finally, what I owe to my long-time companion Steve Maddy is too complex and varied to be summarized here. I am grateful to all these people and offer my heartfelt thanks. Also to Angela Blackburn and Frances Morphy of Oxford University Press.

Much of this book is based on a series of articles (Maddy 1980, 1981, 1984a, 1988a,b, forthcoming a, b) the preparation of which was supported, at various times, by the American Association of University Women, the University of Notre Dame, the National Endowment for the Humanities, the National Science Foundation, and the University of Illinois at Chicago. The original publishers kindly granted advance permission to reproduce material from these pieces; in the end, only parts of (forthcoming a) (in chapter 5, sections 1 and 2) and (forthcoming b) (in chapter 1, section 4) actually survived, so I am particularly obliged to Kluwer Academic Publishers and the Association for Symbolic Logic. Preparation of the final draft was supported by National Science Foundation Grant DIR-8807103, a University of California President’s Research Fellowship in the Humanities, and the University of California at Irvine. The help of all these institutions is hereby gratefully acknowledged.

Finally, I feel compelled to add a personal note on sexist language. Some years ago, when I first introduced the ideas behind set theory realism, constructions like ‘the set theoretic realist thinks his entities . . .’ struck me as amusing, but since then I’ve discovered that some readers and editors are legitimately disapproving of this usage. Of the many alternatives available, I’ve chosen one that does the least violence to the standard rhythm, that is, the use of ‘she’
and ‘her’ in place of ‘he’ and ‘his’ in neutral contexts. Some might find this just as politically incorrect as the automatic use of the masculine, but I sincerely doubt that phrasing like ‘when the mathematician proves a theorem, she . . .’ makes anyone tend to forget that some mathematicians are men. So I’ll stick with this policy. To those who find it distracting, I apologize; this is not, after all, a political treatise. At least you have my reasons.

P.M.

Irvine, California
June 1989
This page intentionally left blank
<table>
<thead>
<tr>
<th>CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Realism</td>
</tr>
<tr>
<td>1. Pre-theoretic realism</td>
</tr>
<tr>
<td>2. Realism in philosophy</td>
</tr>
<tr>
<td>3. Realism and truth</td>
</tr>
<tr>
<td>4. Realism in mathematics</td>
</tr>
<tr>
<td>2. Perception and Intuition</td>
</tr>
<tr>
<td>1. What is the question?</td>
</tr>
<tr>
<td>2. Perception</td>
</tr>
<tr>
<td>3. Intuition</td>
</tr>
<tr>
<td>4. Gödelian Platonism</td>
</tr>
<tr>
<td>3. Numbers</td>
</tr>
<tr>
<td>1. What numbers could not be</td>
</tr>
<tr>
<td>2. Numbers as properties</td>
</tr>
<tr>
<td>3. Frege numbers</td>
</tr>
<tr>
<td>4. Axioms</td>
</tr>
<tr>
<td>1. Reals and sets of reals</td>
</tr>
<tr>
<td>2. Axiomatization</td>
</tr>
<tr>
<td>3. Open problems</td>
</tr>
<tr>
<td>4. Competing theories</td>
</tr>
<tr>
<td>5. The challenge</td>
</tr>
<tr>
<td>5. Monism and Beyond</td>
</tr>
<tr>
<td>1. Monism</td>
</tr>
<tr>
<td>2. Field’s nominalism</td>
</tr>
<tr>
<td>3. Structuralism</td>
</tr>
<tr>
<td>4. Summary</td>
</tr>
<tr>
<td>References</td>
</tr>
<tr>
<td>Index</td>
</tr>
</tbody>
</table>
This page intentionally left blank
REALISM

1. Pre-theoretic realism

Of the many odd and various things we believe, few are believed more confidently than the truths of simple mathematics. When asked for an example of a thoroughly dependable fact, many will turn from common sense—'after all, they used to think humans couldn't fly'—from science—'the sun has risen every day so far, but it might fail us tomorrow'—to the security of arithmetic—'but 2 plus 2 is surely 4'.

Yet if mathematical facts are facts, they must be facts about something; if mathematical truths are true, something must make them true. Thus arises the first important question: what is mathematics about? If 2 plus 2 is so definitely 4, what is it that makes this so?

The guileless answer is that 2 + 2 = 4 is a fact about numbers, that there are things called '2' and '4', and an operation called 'plus', and that the result of applying that operation to 2 and itself is 4. '2 + 2 = 4' is true because the things it's about stand in the relation it claims they do. This sort of thinking extends easily to other parts of mathematics: geometry is the study of triangles and spheres; it is the properties of these things that make the statements of geometry true or false; and so on. A view of this sort is often called 'realism'.

Mathematicians, though privy to a wider range of mathematical truths than most of us, often incline to agree with unsullied common sense on the nature of those truths. They see themselves and their colleagues as investigators uncovering the properties of various fascinating districts of mathematical reality: number theorists study the integers, geometers study certain well-behaved spaces, group theorists study groups, set theorists sets, and so on. The very experience of doing mathematics is felt by many to support this position:
The main point in favor of the realistic approach to mathematics is the instinctive certainty of most everybody who has ever tried to solve a problem that he is thinking about ‘real objects’, whether they are sets, numbers, or whatever . . . (Moschovakis (1980), 605)

Realism, then (at first approximation), is the view that mathematics is the science of numbers, sets, functions, etc., just as physical science is the study of ordinary physical objects, astronomical bodies, subatomic particles, and so on. That is, mathematics is about these things, and the way these things are is what makes mathematical statements true or false. This seems a simple and straightforward view. Why should anyone think otherwise?

Alas, when further questions are posed, as they must be, embarrassments arise. What sort of things are numbers, sets, functions, triangles, groups, spaces? Where are they? The standard answer is that they are abstract objects, as opposed to the concrete objects of physical science, and as such, that they are without location in space and time. But this standard answer provokes further, more troubling questions. Our current psychological theory gives the beginnings of a convincing portrait of ourselves as knowers, but it contains no chapter on how we might come to know about things so irrevocably remote from our cognitive machinery. Our knowledge of the physical world, enshrined in the sciences to which realism compares mathematics, begins in simple sense perception. But mathematicians don’t, indeed can’t, observe their abstract objects in this sense. How, then, can we know any mathematics; how can we even succeed in discussing this remote mathematical realm?

Many mathematicians, faced with these awkward questions about what mathematical things are and how we can know about them, react by retreating from realism, denying that mathematical statements are about anything, even denying that they are true: ‘we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say “Mathematics is just a combination of meaningless symbols” . . .’.¹ This formalist position—that mathematics is just a game with symbols—faces formidable obstacles of its own, which I’ll touch on below, but even without these, many mathematicians find it involving them in an uncomfortable form of

¹ Dieudonné, as quoted in Davis and Hersh (1981), 321.
double-think. The same writer continues: ‘Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working on something real’ (Davis and Hersh (1981), 321). Two more mathematicians summarize:

the typical working mathematician is a [realist] on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all. (Davis and Hersh (1981), 321)

Yet this occasional inauthenticity is perhaps less troubling to the practising mathematician than the daunting requirements of a legitimate realist philosophy:

Nevertheless, most attempts to turn these strong [realist] feelings into a coherent foundation for mathematics invariably lead to vague discussions of ‘existence of abstract notions’ which are quite repugnant to a mathematician . . . Contrast with this the relative ease with which formalism can be explained in a precise, elegant and self-consistent manner and you will have the main reason why most mathematicians claim to be formalists (when pressed) while they spend their working hours behaving as if they were completely unabashed realists. (Moschovakis (1980), 605–6)

Mathematicians, after all, have their mathematics to do, and they do it splendidly. Dispositionally suited to a subject in which precisely stated theorems are conclusively proved, they might be expected to prefer a simple and elegant, if ultimately unsatisfying, philosophical position to one that demands the sort of metaphysical and epistemological rough-and-tumble a full-blown realism would require. And it makes no difference to their practice, as long as double-think is acceptable.

But to the philosopher, double-think is not acceptable. If the very experience of doing mathematics, and other factors, soon to be discussed, favour realism, the philosopher of mathematics must either produce a suitable philosophical version of that position, or explain away, convincingly, its attractions. My goal here will be to do the first, to develop and defend a version of the mathematician’s pre-philosophical attitude.

Rather than attempt to treat all of mathematics, to bring the project
down to more manageable size, I'll concentrate here on the mathematical theory of sets.\textsuperscript{2} I've made this choice for several reasons, among them the fact that, in some sense, set theory forms a foundation for the rest of mathematics. Technically, this means that any object of mathematical study can be taken to be a set, and that the standard, classical theorems about it can then be proved from the axioms of set theory.\textsuperscript{3}

Striking as this technical fact may be, the average algebraist or geometer loses little time over set theory. But this doesn't mean that set theory has no practical relevance to these subjects. When mathematicians from a field outside set theory are unusually frustrated by some recalcitrant open problem, the question arises whether its solution might require some strong assumption heretofore unfamiliar within that field. At this point, practitioners fall back on the idea that the objects of their study are ultimately sets and ask, within set theory, whether more esoteric axioms or principles might be relevant. Given that the customary axioms of set theory don't even settle all questions about sets,\textsuperscript{4} it might even turn out that this particular open problem is unsolvable on the basis of these most basic mathematical assumptions, that entirely new set theoretic assumptions must be invoked.\textsuperscript{5} In this sense, then, set theory is the ultimate court of appeal on questions of what mathematical things there are, that is to say, on what philosophers call the 'ontology' of mathematics.\textsuperscript{6}

Philosophically, however, this ontological reduction of mathematics to set theory has sometimes been taken to have more dramatic consequences, for example that the entire philosophical foundation of any branch of mathematics is reducible to that of set theory. In this sense, comparable to implausibly strong versions of

\textsuperscript{2} A set is a collection of objects. Among the many good introductions to the mathematical theory of these simple entities, I recommend Enderton (1977).

\textsuperscript{3} See e.g. the reduction of arithmetic and real number theory to set theory in Enderton (1977), chs. 4 and 5. There are some exceptions to the rule that all mathematical objects can be thought of as sets—e.g. proper classes and large categories—but I will ignore these cases for the time being.

\textsuperscript{4} Some details and philosophical consequences of this situation are the subject of ch. 4.

\textsuperscript{5} Eklof and Mekler (forthcoming) give a survey of algebraic examples, and Moschovakis (1980) does the same for parts of analysis.

\textsuperscript{6} In philosophical parlance, 'ontology', the study of what there is, is opposed to 'epistemology', the study of how we come to know what we do about the world. I will use the word 'metaphysics' more or less as a synonym for 'ontology'. 
the thesis that physics is basic to the natural sciences, I think the claim that set theory is foundational cannot be correct. Even if the objects of, say, algebra are ultimately sets, set theory itself does not call attention to their algebraic properties, nor are its methods suitable for approaching algebraic concerns. We shouldn’t expect the methodology or epistemology of algebra to be identical to that of set theory any more than we expect the biologist’s or the botanist’s basic notions and techniques to be identical to those of the physicist. But again, this methodological independence of the branches of mathematics from set theory does not mean there must be mathematical entities other than sets any more than the methodological independence of psychology or chemistry from physics means there must be non-physical minds or chemistons.

But little hangs on this assessment of the nature of set theory’s foundational role. Even if set theory is no more than one among many branches of mathematics, it is deserving of philosophical scrutiny. Indeed, even as one branch among many, contemporary set theory is of special philosophical interest because it throws into clear relief a difficult and important philosophical problem that challenges many traditional attitudes toward mathematics in general. I will raise this problem in Chapter 4.

Finally, it is impossible to divorce set theory from its attendant disciplines of number theory and analysis. These two fields and their relationship to the theory of sets will form a recurring theme in what follows, especially in Chapters 3 and 4.

2. Realism in philosophy

So far, I’ve been using the key term ‘realism’ loosely, without clear definition. This may do in pre-philosophical discussion, but from

---

7 This view is called ‘physicalism’. I’ll come back to it in ch. 5, sect. 1, below.
8 There was a time when the peculiarities of biological science led practitioners to vitalism, the assumption that a living organism contains a non-physical component or aspect for whose behaviour no physical account can be given. Nowadays, this idea is discredited—simply because it proved scientifically sterile—and, as far as I know, no one ever urged the acceptance of ‘chemistons’. Today, psychology is the special science that most often lays claim to a non-physical subject matter, but as suggested in the text, it seems to me that a purely physical ontology is compatible with the most extreme methodological independence. For discussion, see Fodor (1975), 9–26.
now on I will try to be more precise. This doesn’t mean I’ll succeed
in defining the term exactly, but at least I’ll narrow the field
somewhat, I hope helpfully. Let me begin by reviewing some
traditional uses of the term in philosophy.

One of the most basic ontological debates in philosophy concerns
the existence of what common sense takes to be the fundamental
furniture of the world: stones and trees, tables and chairs, medium-
sized physical objects. Realism in this context, often called
‘common-sense realism’, affirms that these familiar macroscopic
things do in fact exist. But it is not enough for the realist to insist
that there are stones and trees and such like, for in this much the
idealist could agree, all the while assuming that a stone is a mental
construct of some sort, say a bundle of experiences. However, such
an idealist, like the Bishop Berkeley, will have serious trouble
agreeing with the realist that stones can exist without being
perceived. Thus the common-sense realist can state her position in
a way that rules out idealism by claiming that stones etc. exist, and
that their existence is non-mental, that they are as they are
independently of our ability to know about them, that their
existence is, in a word, ‘objective’.

A more recent opponent of the common-sense realist uses a more
devious technique. The phenomenalist hopes to say exactly what
the realist says while systematically reinterpreting each and every
physical object claim into a statement about what she calls ‘sense
data’, or really, into statements about possible sense data. For
example, my overcoat exists in the closet though unperceived
because part of the translation of ‘the overcoat is in the closet’ is
something like ‘if I were in the closet and the light were on, then I’d
have an overcoat-like experience’, which is, presumably, true.
Physical objects are not taken to consist of ideas, as with Berkeley,
but physical object statements are taken to mean something other
than what we ordinarily take them to mean.

9 Berkeley’s notorious solution was to suppose that God is perceiving the object
even when we aren’t; indeed he uses this as a novel argument for the existence of
God. See, e.g. Berkeley (1713), 211–13, 230–1. It’s worth noting, however, that in
earlier work, Berkeley (1710 §§ 3, 58–9) includes a ‘counterfactual’ analysis that
prefigures the Millian phenomenализm described in the next paragraph.

10 This idea took shape in Mill (1865), ch. 11 and its appendix, and was
developed in the form described here by the logical positivists. See Ayer (1946),
63–8.